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RUZSA'S PROBLEM ON SETS OF RECURRENCE

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ABSTRACT

The main purpose of this paper is to prove the existence of Poincaré sequences of integers which are not van der Corput sets. This problem was considered in I. Ruzsa's expository article [R_1] (1982–83) on correlative and intersective sets. Thus the existence is shown of a positive non-continuous measure μ on the circle which Fourier transform vanishes on a set of recurrence, i.e. $S = \{n \in \mathbf{Z};\ \hat{\mu}(n) = 0\}$ is a set of recurrence but not a van der Corput set. The method is constructive and involves some combinatorial considerations. In fact, we prove that the generic density condition for both properties are the same.

1. Definitions and preliminairies

Given a subset S of the integers $\mathbf{N}$, let

$$D^*(S) = \overline{\lim_{N\to\infty}} \frac{|S \cap [1, N]|}{N}$$

be the upper density of S.

In what follows, we will recall some definitions from [F] and [R1] (cf. also [B–M]).

A subset Λ of the positive integers is called a *Poincaré* or *recurrent* set (P) provided whenever $(X, \mathscr{B}, \mu, T)$ is a dynamical system and A a measurable set of positive measure, then

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$$\mu(T^{-m}A \cap A) > 0 \quad \text{for some } m \in \Lambda$$

(compare with the classical Poincaré recurrence theorem). It follows from [F] and [B–M] that property (P) is equivalent to *intersectivity* (cf. [R1]), meaning that

$$\Lambda \cap (S - S) \neq \emptyset$$

whenever $D^*(S) > 0$.

The set Λ is called a *van der Corput set* (v.d.C) provided for sequences of real numbers $(u_n)_{n \geqq 0}$, the uniform distribution mod 1 of each different sequence $v_n = u_{n+h} - u_n$ for $h \in \Lambda$ implies the uniform distribution of the sequence (u_n) itself. (Compare with the van der Corput criterion for uniform distribution.) There are several equivalent formulations of this property, due to Kamae–Mendès France [K–M] and Rusza [R1] and which we list now:

(1.1) If $\mu \in M_+(\mathbf{T})$ is a positive measure on the circle and

$$\hat{\mu}(n) = \int e^{-in\theta} \mu(d\theta) = 0 \quad \text{for } n \in \Lambda$$

then μ is continuous.

(1.2) Λ is *correlative*, meaning that whenever (y_n) is a sequence of complex numbers satisfying

$$\sum_{n \leqq x} |y_n|^2 = O(x),$$

$$\sum_{n \leqq x} y_{n+k}\bar{y}_n = o(x) \quad (k \in \Lambda) \quad \text{(correlation condition)}$$

then

$$\sum_{n \leqq x} y_n = o(x).$$

(1.3) Given $\varepsilon > 0$, there is a polynomial $P(x) = \sum_{n \in \Lambda \cup \{0\}} a_n \cos nx$ with $a_n \in \mathbf{R}$ satisfying

$$P(x) \geqq 0, \quad P(0) = 1, \quad a_0 \leqq \varepsilon.$$

It was observed in [K–M] that van der Corput sets are Poincaré sets. Our purpose is to show that the converse implication is false. Thus

THEOREM. *There is an intersective set which is not correlative* (cf. [R1]).

In the next section, it is shown that no counterexample can be obtained by statistical considerations, since the generic density conditions for both properties are the same. This fact was perhaps already observed earlier by others, but we include a proof here since it shows the necessity of a more deterministic approach.

Let us recall two more properties. The first is related to the Bohr topology (cf. [R1]). For a finite set $V = (v_1, \ldots, v_k)$ of real numbers, write for $\varepsilon > 0$

$$S(v, \varepsilon) = \{n; \| v_j n \| < \varepsilon \text{ for } j = 1, \ldots, k\}$$

where $\| x \|$ stands for the distance of x to the nearest integer. Call $\Lambda \subset N$ *approximative* provided

$$\Lambda \cap S(v, \varepsilon) \neq \emptyset$$

for any finite v and $\varepsilon > 0$.

As observed in [R1], this means that $v_1, \ldots, v_k$ can be simultaneously approximated by rationals having a common denominator from Λ.

Finally, call $\Lambda \subset \mathbf{N}$ an *FC^+-set* (forcing continuity for positive measures) provided

$$\mu \in M_+(\mathbf{T}), \quad \hat{\mu}(n) \to 0 \quad \text{on } \Lambda \Rightarrow \mu \quad \text{is continuous.}$$

The following implications hold:

$$(1.4) \qquad (\mathrm{FC}^+) \Rightarrow (\mathrm{v.d.C}) \Rightarrow (\mathrm{P}) \Rightarrow (\text{approximative}).$$

Notice that the class of sets which do not satisfy one of the properties listed in (1.4) is closed under finite union.

There are also quantitative versions of the notions discussed above (see [R1]) of interest in the study of concrete examples.

A first step in proving the theorem, in fact containing the main idea, will be to construct a Poincaré set which is not (FC^+) (see Section 3). The actual vanishing property for the Fourier transform is then achieved by modifying the previous construction, modifications mainly of a technical nature (Sections 4, 5, 6).

We also refer the reader to Y. Peres' thesis for more details on properties (P) and (v.d.C).

2. Discussion of the generic case

The result obtained in this section is a variant of statistical verifications of certain harmonic analysis properties for random subsets of **Z** with prescribed density (compare with [K1], [K2], for instance).

PROPOSITION 2.1. *Let* $\mathbf{N} = \bigcup I_k$ *be a partition of the integers in intervals, say* $I_k = [2^{2^k}, 2^{2^{k+1}}]$. *Choose for each* k *a random subset* Λ_k *of* I_k, $|\Lambda_k| = N_k$, *assigning to each element of* I_k *the same probability* δ_k. *Let* $\Lambda = \bigcup_{k=1}^{\infty} \Lambda_k$. *Then almost surely*

(1) *If* $\overline{\lim}_k 2^{-k} N_k < \infty$, *then* Λ *is not approximative.*

(2) *If* $\overline{\lim}_k 2^{-k} N_k = \infty$, *then* Λ *is an* FC^+*-set.*

In order to prove (2), we use the following lemma (compare with [K–M] and (1.3)). We denote $A(\mathbf{T})$ the space of absolutely convergent Fourier series.

LEMMA 2.2. *Let* $\Lambda \subset \mathbf{Z}$, *for which there exists a sequence* $(\varphi_n)_{n=1,2,\ldots}$ *in* $A(\mathbf{T})$ *satisfying the conditions*

(i) $\hat{\varphi}_n$ *is supported by* Λ,

(ii) $\sup \| \varphi_n \|_{A(\mathbf{T})} < \infty$,

(iii) $\varphi_n(0) = 1$,

(iv) $\varphi_n(x) \to 0$ *for* $x \in \mathbf{T}$, $x \neq 0$.

Then Λ *is an* FC^+*-set.*

PROOF. Let $\mu \in M_+(\mathbf{T})$. Since (φ_n) is uniformly bounded, (iv) implies

$$\lim_{n\to\infty} \int_{\mathbf{T}} \varphi_n(x)\mu(dx) = \mu(\{0\}). \tag{2.3}$$

Since $\lim_{n\to\infty} \hat{\varphi}_n(k) = 0$ for each k, it follows from (ii) that

$$\overline{\lim_n} \left| \sum \hat{\varphi}_n(k)\hat{\mu}(k) \right| \leqq c \overline{\lim_{k\in\Lambda}} \, |\hat{\mu}(k)|. \tag{2.4}$$

It follows that $\mu(\{0\}) = 0$ if $\hat{\mu} \to 0$ on Λ.

PROOF OF PROPOSITION 2.1(2). Assume $\overline{\lim}_k 2^{-k} N_k = \infty$. Fix k and consider independent (0,1)-valued selectors $(\xi_n)_{n\in I_k}$ of mean δ, $\delta \cdot |I_k| = N_k$. Define the random function

$$\varphi_\omega(x) = \frac{1}{N_k} \sum_{n\in I_k} \xi_n(\omega) e^{inx}.$$

Thus

$$\int \| \varphi_\omega \|_A d\omega = 1 = \int \varphi_\omega(0) d\omega. \tag{2.5}$$

Also

$$|\varphi_\omega(x)| \leqq \frac{\delta}{N_k} \left| \sum_{n \in I_k} e^{inx} \right| + \frac{1}{N_k} \left| \sum_{I_k} (\xi_n - \delta) e^{inx} \right| . \tag{2.6}$$

The first term is bounded by $|I_k|^{-1} |1 - e^{ix}|^{-1}$, which is small for x not too close to 0. For the second term in (2.6) we apply standard probabilistic estimates to get (since the $\xi_n - \delta$ are independent of mean 0)

$$\int \left\| \sum_{I_k} (\xi_n(\omega) - \delta) e^{inx} \right\|_\infty d\omega$$

$$\leqq c(\log |I_k|)^{1/2} \int \left(\sum_{I_k} \xi_n(\omega)^2 \right)^{1/2} d\omega \leqq c(2^k N_k)^{1/2}.$$

This will be $o(N_k)$ provided $2^{-k} N_k \rightarrow \infty$. Thus it results from (2.6) that given $\tau > 0$, for an appropriate k

$$\int \sup_{|x| > \tau} |\varphi_\omega(x)| d\omega < \tau. \tag{2.7}$$

Using (2.5), (2.7) it is now easy to satisfy the hypothesis of Lemma 2.2.

Assertion (1) of Proposition 2.1 is related to an observation of Katznelson and Malliavin that the random sets obtained under hypothesis (1) are almost surely Helson.

PROOF OF PROPOSITION 2.1(1). By considerations of finite union, it suffices to show that for some $\tau > 0$, if

$$\overline{\lim}\, 2^{-k} N_k < \tau \tag{2.8}$$

then almost surely

$$\sup_{x \in \mathbf{T}} \overline{\lim_{n \in \Lambda}} |1 - e^{inx}| > 0. \tag{2.9}$$

Let φ be a positive function on $\mathbf{T}$ satisfying the conditions

$$\begin{cases} \varphi(x) = 0 & \text{if } |1 - e^{ix}| < \frac{1}{4}, \\ \hat{\varphi}(0) = 1, & \\ \| \varphi \|_A < 20, & \end{cases}$$

The sets Λ_k are obtained by choosing at random N_k elements in the interval I_k. Notice that in the limit this choice is almost surely not repetitive. Notice also that (2.9) will hold if for some k and all $l > k$, the function

$$\prod_{n \in \Lambda_k \cup \cdots \cup \Lambda_l} \varphi(nx) \neq 0. \tag{2.10}$$

Indeed, (2.10) will give a point x_l so that

$$|1 - e^{inx_l}| \geqq \tfrac{1}{4} \quad \text{for } n \in \Lambda_k \cup \cdots \cup \Lambda_l.$$

By compactness, there is then a point $x \in \pi$ fulfilling

$$|1 - e^{inx}| \geqq \tfrac{1}{4} \quad \text{if } n \in \bigcup_{l \geqq k} \Lambda_l.$$

Thus, by the Borel–Cantelli lemma, it remains to show that for k large enough and any $l > k$, with probability at least $\frac{1}{2}$

$$\int \Big\{ \prod_{n \in \Lambda_k \cup \cdots \cup \Lambda_l} \varphi(nx) \Big\} dx > 0. \tag{2.11}$$

In fact, we show that

$$\left| 1 - \int \Big\{ \prod_{n \in \Lambda_k \cup \cdots \cup \Lambda_l} \varphi(nx) \Big\} dx \right| < \tfrac{1}{2} \tag{2.12}$$

with probability at least $\frac{1}{2}$.

The argument is based on the following observation. If f is a function in $A(\mathbf{T})$, then

$$\frac{1}{|I_j|} \sum_{n \in I_j} \left| \hat{f}(0) - \int f(x) \varphi(nx) dx \right| \leqq \frac{20}{|I_j|} \| f \|_A \tag{2.13}$$

by hypothesis on φ and since

$$\hat{f}(0) - \int f(x) \varphi(nx) dx = \sum_{\substack{k \in \mathbf{Z} \\ k \neq 0}} \hat{\varphi}(k) \hat{f}(-nk).$$

The claim (2.12) then easily follows by iteration of (2.13), writing a telescopic sum. We are led to the condition

$$\sum_{j=k}^{l}\sum_{s=0}^{N_j-1}\|\varphi\|_A^{N_k+\cdots+N_{j-1}+s}\frac{20}{|I_j|}<\frac{1}{20}$$

or

$$\sum_{j=k}^{l}20^{2\tau 2^j}2^{-2^j}<\frac{1}{400},$$

clearly satisfied for appropriate τ.

This proves Proposition 2.1. Thus for random subsets of the integers, the generic density condition for properties (FC^+), (v.d.C), (P) and density in the Bohr compactification are the same.

3. Construction of a recurrent set which is not (FC^+)

For $t\in T$, let δ_t be the Dirac measure.

PROPOSITION 3.1. *Define the infinite convolution*

$$\nu=\mathop{*}_{j=1}^{\infty}\left[\tfrac{1}{2}(\delta_{2\pi/j!}+\delta_{-2\pi/j!})\right]$$

and let $\mu=\delta_0+\nu$. *Let* $\alpha:\mathbf{N}\to\mathbf{R}_+$ *satisfy* $\lim_{n\to\infty}\alpha(n)=\infty$. *Then the set*

$$\Lambda=\bigcup_N\{1\leqq n\leqq N!:|\hat{\mu}(n)|<\alpha(N)/N\}$$

is recurrent.

Obviously Λ is not (FC^+) provided $\lim_{n\to\infty}(\alpha(n)/n)=0$.

Proposition 3.1 is clearly a consequence of

LEMMA 3.2. *Let* A *be a subset of* $\{1,2,\ldots,J!\}$, $|A|>cJ!$. *Then*

$$\min_{n\in A-A}|\hat{\mu}(n)|<M(c)/J. \tag{3.3}$$

Denote $\Omega_j=\{0,1,\ldots,j\}$ and $\Omega=\Omega_1\times\Omega_2\times\cdots\times\Omega_{J-1}$. The representation

$$n = \sum_{j=1}^{J-1} q_j j! \qquad (0 \leqq q_j \leqq j) \tag{3.4}$$

defines a one-to-one map from $\{1, 2, \ldots, J! - 1\}$ to Ω. Denote by ν the normalized counting measure on Ω (= product measure). Let A be a subset of $\{1, \ldots, J! - 1\}$, $|A| > cJ!$ and $\tilde{A} \subset \Omega$ the image of A under the mapping considered above. Thus $\nu(\tilde{A}) > c$. The following combinatorial lemma will be used.

LEMMA 3.5. *Let $B \subset \Omega$, $\nu(B) > c$. Then there is a pair of points x, x' in B and an integer $cJ < s < J$ such that*

$$x_1 = x'_1, \ldots, x_{s-1} = x'_{s-1},$$

$$x_s = 0, \quad x'_s = [s/2],$$

$$|x_{s+1} - x'_{s+1}| \leqq 2, \quad \ldots, \quad |x_{J-1} - x'_{J-1}| \leqq 2.$$

PROOF. Perform the following construction:

$B_J = B,$

$B_{J-1} = \{t \in \Omega \mid \text{there is } t' \in B_J \text{ with } t_1 = t'_1, \ldots, t_{J-2} = t'_{J-2}, |t_{J-1} - t'_{J-1}| \leqq 1\},$

$B_{J-2} = \{t \in \Omega \mid \text{there is } t' \in B_{J-1} \text{ with } t_1 = t'_1, \ldots, t_{J-3} = t'_{J-3},$

$\qquad |t_{J-2} - t'_{J-2}| \leqq 1, t_{J-1} = t'_{J-1}\},$

$\vdots$

$B_s = \{t \in \Omega \mid \text{there is } t' \in B_{s+1} \text{ with } t_1 = t'_1, \ldots, t_{s-1} = t'_{s-1},$

$\qquad |t_s - t'_s| \leqq 1, t_{s+1} = t'_{s+1}, \cdots\},$

$\vdots$

Thus each element of B_s can be perturbed in the s-th coordinate by at most one unit to become an element of B_{s+1}. Perturbing the $J - s$ last coordinates, an element of B is obtained.

Denote by π_s the projection on the coordinates $1, \ldots, s-1, s+1, \ldots, J-1$. If for each $x \in \pi_s(B_{s+1})$ there is $t \in \Omega \setminus B_{s+1}$ with $\pi_s(t) = x$, then clearly

$$\nu(B_s \setminus B_{s+1}) \geqq \frac{1}{s} \nu_s(B_{s+1})) \geqq \frac{1}{s} \nu(B_{s+1}),$$

$$\nu(B_s) \geqq \left(1 + \frac{1}{s}\right) \nu(B_{s+1}). \tag{3.6}$$

Fixing $1 < \bar{J} < J$, (3.6) and the fact that $\Pi_{s=\bar{J}}^{J}(1 + 1/s) = J/\bar{J}$, imply the existence of some $s > \nu(B) \cdot J$ and a point $x \in \pi_{\bar{s}}(B_{s+1})$ satisfying

$$t \in \Omega, \qquad \pi_{\bar{s}}(t) = x \Rightarrow t \in B_{s+1}.$$

Thus, by construction, one may find for each $p \in \{0, 1, \ldots, s\}$ an element

$$(x_1, \ldots, x_{s-1}, p, x'_{s+1}, \ldots, x'_{J-1})$$

in B, where $|x'_{s+1} - x_{s+1}| \leqq 1, \ldots, |x'_{J-1} - x_{J-1}| \leqq 1$. The lemma follows.

Proof of Lemma 3.2. Applying Lemma 3.5 to the set $\tilde{A}$, a pair of elements

$$n = \sum_{j=1}^{J-1} x_j j!, \quad n' = \sum_{j=1}^{J-1} x'_j j!$$

in A is obtained, where (x_j), (x'_j) fulfill the condition of Lemma 3.5. Thus

$$m \equiv n' - n = [s/2]s! + (x'_{s+1} - x_{s+1})(s+1)! + \cdots + (x'_{J-1} - x_{J-1})(J-1)!$$

is in the difference set $A - A$. By definition of μ

$$\hat{\mu}(m) = 1 + \prod_{j=1}^{\infty} \cos 2\pi \frac{m}{j!} = 1 + \prod_{j=s+1}^{\infty} \cos 2\pi \frac{m}{j!}.$$

Hence

$$\begin{aligned} |\hat{\mu}(m)| &\leqq \left| 1 + \cos 2\pi \frac{m}{(s+1)!} \right| + \sum_{j>s+1} \left| 1 - \cos 2\pi \frac{m}{j!} \right| \\ &= 2\cos^2 \pi \frac{m}{(s+1)!} + 2 \sum_{j>s+1} \sin^2 \pi \frac{m}{j!} \end{aligned} \tag{3.7}$$

where

$$\left| \cos \pi \frac{m}{(s+1)!} \right| = \left| \cos \pi \left[\frac{s}{2}\right] \frac{1}{s+1} \right| = O\left(\frac{1}{s}\right)$$

and since $|x_{s+1} - x'_{s+1}| \leqq 2, \ldots, |x_{J-1} - x'_{J-1}| \leqq 2$, for $j \geqq s+2$,

$$\left|\sin \pi \frac{m}{j!}\right|$$

$$= \left|\sin \pi \left\{\left[\frac{s}{2}\right]\frac{s!}{j!} + (x'_{s+1} - x_{s+1})\frac{(s+1)!}{j!} + \cdots + (x'_{j-1} - x_{j-1})\frac{(j-1)!}{j!}\right\}\right|$$

$$= O\left(\frac{1}{j}\right).$$

Substitution in (3.7) thus yield

$$|\hat{\mu}(m)| \leqq \text{const}\left(\frac{1}{s^2} + \sum_{j>s}\frac{1}{j^2}\right) \sim \frac{1}{s} < \frac{1}{cJ}$$

using the lower estimate on s given by Lemma 3.5. This completes the proof of Lemma 3.2.

The purpose of the next sections is to modify previous construction in order to obtain a (P)-set on which $\hat{\mu}$ actually vanishes.

4. Reduction to a local problem

Assume for each j positive integers $n_j < \frac{1}{10}N_j$ given and a trigonometric polynomial p_j satisfying the following conditions:

(4.1) $$p_j \geqq 0, \quad \hat{p}_j(0) = 1,$$

(4.2) $$\text{supp}\, \hat{p}_j \subset [\tfrac{1}{4}N_j, \tfrac{1}{4}N_j],$$

(4.3) If $A \subset [0, n_j]$, $|A| > \frac{1}{j} n_j$ then $\hat{p}_j(m) = -\frac{1}{2}$ for some $m \in A - A$.

LEMMA 4.4. *Under the conditions* (4.1), (4.2), (4.3), *there is a positive measure* μ, $\mu(\{0\}) = 1$, *such that* $\hat{\mu}$ *vanishes on some set of recurrence.*

PROOF. Define

$$q_j(t) = p_1(t)p_2(N_1t)p_3(N_1N_2t)\cdots p_j(N_1\cdots N_{j-1}t)$$

for which

$$\text{supp}\, \hat{q}_j \subset [-\tfrac{1}{3}N_1\cdots N_j, \tfrac{1}{3}N_1\cdots N_j].$$

Thus since

$$q_j(t) = q_{j-1}(t)p_j(N_1 \cdots N_{j-1}t)$$

we have

$$\int q_j(t)dt = \left(\int q_{j-1}\right)\left(\int p_j\right) = 1. \tag{4.5}$$

Also for $|n| < \frac{1}{2}N_1 \cdots N_{j-1}$,

$$\hat{q}_j(n) = \hat{q}_{j-1}(n). \tag{4.6}$$

If $n = N_1 \cdots N_{j-1}m$, then

$$\hat{q}_j(n) = \hat{p}_j(m). \tag{4.7}$$

Let ν be the weak*-limit of the sequence $\{q_j\}$ in $M(\mathrm{T})$. By (4.5), (4.6), (4.7)

$$\| \nu \| = 1, \tag{4.8}$$

$$\hat{\nu}(n) = \hat{p}_j(m) \quad \text{if } n = N_1 \cdots N_{j-1}m \quad \text{if } m < \tfrac{1}{2}N_j. \tag{4.9}$$

Define

$$\mu = \delta_0 + 2\nu. \tag{4.10}$$

Take $S \subset \mathbf{Z}$, $D^*(S) > \varepsilon > 1/j$. Since the class $\{A - A \mid A \subset S,\ A \text{ finite}\}$ is homogeneous in the sense of [R2] there is a subset A_1 of $[0, N_1 \cdots N_{j-1}n_j]$ for which

$$A_1 - A_1 \subset S - S, \tag{4.11}$$

$$|A_1| > \varepsilon N_1 \cdots N_{j-1}n_j. \tag{4.12}$$

Hence there is $A \subset \{1, \ldots, n_j\}$ satisfying

$$\begin{aligned} &(A - A)N_1 \cdots N_{j-1} \subset A_1 - A_1, \\ &|A| > \varepsilon n_j. \end{aligned} \tag{4.13}$$

By (4.3) $\hat{p}_j(m) = -\frac{1}{2}$ for some $m \in A - A$. Thus if $n = N_1 \cdots N_{j-1}m$, then $n \in S - S$ by (4.13), (4.11) and since $|m| < \frac{1}{2}N_j$, by (4.9)

$$\hat{\mu}(n) = 1 - 2\hat{\nu}(n) = 1 + 2\hat{p}_j(m) = 0.$$

This proves the lemma.

Fixing an integer n and $\varepsilon > 0$, our purpose will be to construct a positive measure μ, $\| \mu \| = \frac{3}{4}$, satisfying

$$\hat{\mu}(m) = -\tfrac{1}{2} \quad \text{for some } m \in A - A$$

whenever

$$(*) \qquad A \subset [0, n], \quad |A| > \varepsilon n.$$

Given μ, let for some N

$$p_1 = (\mu * F_N) + \{\mu - (\mu * F_N)] * D_n$$

where $F_N = N$-Féjer kernel and $D_n = n$-Dirichlet kernel. For N large enough, we may ensure that

$$\| [\mu - (\mu * F_N)] * D_n \|_\infty < \tfrac{1}{4}.$$

Thus $p = p_1 + \frac{1}{4}$ is a positive polynomial and

$$\hat{p}(m) = \hat{p}_1(m) = \hat{\mu}(m) \quad \text{for } |m| \leqq n, \quad m \neq 0.$$

It is now clear how to get from $(*)$ a sequence $\{p_j\}$ satisfying the conditions of Lemma 4.4.

5. Construction of certain measures

Fix an integer N and consider the basic measure

$$(5.1) \qquad \sigma = \tfrac{1}{2}(\delta_{2\pi/N} + \delta_{-2\pi/N})$$

with transform

$$\hat{\sigma}(n) = \cos 2\pi \frac{n}{N}.$$

In this section we construct a perturbation σ_1 of σ satisfying the following conditions.

LEMMA 5.1. *Given R and a number L, there is a positive measure σ_1 such that σ_1 is supported by the N-th roots of unity and*

$$(5.2) \qquad \hat{\sigma}_1(n) = 1 \quad \text{if } L \leqq |n| \leqq RL,$$

$$(5.3) \qquad \hat{\sigma}_1(n) = -1 \quad \text{if } |N/2 - n| \leqq RL,$$

(5.4) $$\| \sigma - \sigma_1 \|_{M(\mathbf{T})} \leqq c(R)(L/N)^2.$$

PROOF. We assume $RL \ll N$, N even. Consider the polynomial

$$q_1(t) = \sum_{|n| \leqq RL} \left[1 - \cos \frac{2\pi}{N} (RL - |n|) \right] e^{int}.$$

Since the function

$$\begin{cases} 1 - \cos \dfrac{2\pi}{N} (RL - n) & \text{if } 0 \leqq n \leqq RL \\ 0 & \text{if } n > RL \end{cases}$$

is nonnegative, decreasing and convex, q_1 is positive. Hence

(5.5) $$\| q_1 \|_1 = 1 - \cos \frac{2\pi}{N} RL < 10 \left(\frac{RL}{N} \right)^2.$$

Define

$$q_2 = q_1 * F_L; \quad F_L(t) = \sum_{|n| \leqq L} \frac{L - |n|}{L} e^{int} = \text{Féjer kernel.}$$

Then

(5.6) $$\operatorname{supp} \hat{q}_2 \subset [-L, L],$$

(5.7) $$q_2 \geqq 0, \quad \| q_2 \|_1 \leqq 10 \left(\frac{RL}{N} \right)^2,$$

(5.8) $$q_1 \leqq 10 R q_2,$$

the last property following from the fact that $\operatorname{supp} \hat{q}_1 \subset [-RL, RL]$.

Next define the polynomial

$$q_3(t) = 40 R q_2(t) + [2 \cos RLt - \cos(\tfrac{1}{2}N - RL)\, t - \cos(\tfrac{1}{2}N + RL) t] q_1(t).$$

By (5.8), (5.7)

(5.9) $$q_3 \geqq 0, \quad \| q_3 \|_1 \leqq 500 R (RL/N)^2.$$

If $L < n < RL$, then

$$\hat{q}_3(n) = \hat{q}_1(n - RL) = 1 - \cos\frac{2\pi}{N}n. \tag{5.10}$$

If $|\frac{1}{2}N - n| < RL$, then

$$\begin{aligned} 2\hat{q}_3(n) &= -\hat{q}_1(n - \tfrac{1}{2}N + RL) - \hat{q}_1(n - \tfrac{1}{2}N - RL) \\ &= -\hat{q}_1(RL - |n - \tfrac{1}{2}N|) = -1 - \cos\frac{2\pi}{N}n. \end{aligned} \tag{5.11}$$

Finally consider the positive measure

$$\sigma_1 = \sigma + \frac{1}{N}q_3 \cdot \sum_{k=0}^{N-1} \delta_{2\pi k/N}$$

for which, by (5.9),

$$\| \sigma_1 - \sigma \|_{M(\mathbf{T})} \leqq \frac{1}{N}\sum_{k=0}^{N-1} q_3\left(2\pi\frac{k}{N}\right) = \| q_3 \|_1 < 500R^3\left(\frac{L}{N}\right)^2$$

while

$$\hat{\sigma}_1(n) = \hat{\sigma}(n) + \frac{1}{N}\sum_{k=0}^{N-1}(q_3e^{-int})\big|_{t=2\pi k/N} = \cos\frac{2\pi}{N}n + \sum\{\hat{q}_3(m) \mid m - n \in N\mathbf{Z}\}$$

and (5.2), resp. (5.3), follow from (5.10), resp. (5.11), as is easily verified.

6. Proof of existence of a (P)-set which is not (v.d.C)

Our aim is to satisfy (*) in Section 4. We will use arguments similar to those of Section 3 and the measures constructed in the previous section. Take n of the form Q^P, Q even. Fix an integer R. Use the representation

$$m = \sum_{j=0}^{P-1} q_j Q^j \qquad (0 \leqq q_j < Q) \tag{6.1}$$

to get a one-to-one map from $[0, n-1]$ into $\Omega \equiv \{0, 1, \ldots, Q-1\}^P$. Denote ν the normalized counting measure on Ω. Identifying $\{0, 1, \ldots, Q-1\}$ with the cyclic group $\mathbf{Z}/Q\mathbf{Z}$, denote θ the coordinate-wise shift acting on Ω. By Lemma 5.1, we get for each j a positive measure σ_j on $\mathbf{T}$ satisfying the conditions

$$\hat{\sigma}_j(m) = 1 \quad \text{if } Q^j \leqq |m| \leqq RQ^j, \tag{6.2}$$

(6.3) $$\hat{\sigma}_j(m) = -1 \quad \text{if } |Q^{j+1}/2 - m| \leqq RQ^j$$

$(L = Q^j, N = Q^{j+1})$. Moreover

(6.4) $$\| \sigma_j \|_{M(\mathrm{T})} \leqq 1 + C(R)\frac{1}{Q^2}$$

and σ_j is supported by the Q^{j+1}-roots of unity, implying Q^{j+1}-periodicity of $\hat{\sigma}_j$. Define

$$\nu = \sigma_0 * \sigma_1 * \cdots * \sigma_{P-1}.$$

Hence by (6.4)

(6.5) $$\| \nu \|_{M(\mathrm{T})} \leqq 1 + C(R)\frac{P}{Q^2}.$$

Let A be a subset of $[0, n-1]$, $|A| > \varepsilon n$. Let $\tilde{A} \subset \Omega$ be its image under the correspondence mentioned earlier. Thus $\nu(\tilde{A}) > \varepsilon$. Consider next the sets $\tilde{A}$, $\theta(\tilde{A}), \ldots, \theta^R(\tilde{A})$. It is easily seen that for some $4 \leqq r \leqq R - 3$ say, the set

$$B = \tilde{A} \cap \theta^{-r}(\tilde{A})$$

will satisfy $\nu(B) > \varepsilon^2/10$, provided we choose $R > 10/\varepsilon$. (This is the recurrence principle.)

Assuming now

(6.6) $$(1 + 1/Q)^P > 10\varepsilon^{-2}$$

the same combinatorial argument as described in Lemma 5 gives a pair of points x, x' in B:

$$x = (x_1, \ldots, x_{s-1}, 0, x_{s+1}, \ldots, x_p), \quad x' = (x_1, \ldots, x_{s-1}, \tfrac{1}{2}Q, x'_{s+1}, \ldots, x'_p),$$

where

$$|x_j - x'_j| \leqq 2 \quad \text{if } s < j \leqq P.$$

Thus x, $\theta^r(x')$ are both in $\tilde{A}$ corresponding to a pair of points a, a' in A:

$$a = \sum_{j=0}^{P-1} q_j Q^j, \quad a' = \sum_{j=0}^{P-1} q'_j Q^j$$

where

(6.7) $$q'_s - q_s \in \{\tfrac{1}{2}Q + 4, \dots, \tfrac{1}{2}Q + R - 2\},$$

(6.8) $$q'_j - q_j \in \{2, \dots, R-1\} + Q\mathbf{Z} \quad \text{if } j \neq s.$$

Let $m = a' - a$. Thus $m \in A - A$ and we claim that $\hat{\nu}(m) = -1$. Since $\hat{\nu}(m) = \Pi\, \hat{\sigma}_j(m)$, in view of (6.2), (6.3) it suffices to show that

(6.9) $$m \in [\tfrac{1}{2}Q^{s+1} - RQ^s, \tfrac{1}{2}Q^{s+1} + RQ^s] + Q^{s+1}\mathbf{Z}$$

and

(6.10) $$m \in ([-RQ^j, RQ^j] \setminus]-Q^j, Q^j[) + Q^{j+1}\mathbf{Z} \quad \text{if } j \neq s.$$

Clearly, for a fixed j

$$m \in (q'_j - q_j)Q^j + [-Q^j, Q^j] + Q^{j+1}\mathbf{Z}.$$

Therefore (6.9), resp. (6.10), follows from (6.7), resp. (6.8). To satisfy (6.6) take $P = c(\varepsilon)Q$. Then (6.5) implies, together with the condition on R,

$$\| \nu \|_{M(\mathbf{T})} \leqq 1 + c'(\varepsilon)\frac{1}{Q} < \tfrac{3}{2}$$

for Q large enough. Take $\mu = \frac{1}{2}\nu + (\frac{3}{4} - \frac{1}{2}\hat{\nu}(0))$ which will fulfill (*). This completes the proof.

7. Remarks

(1) It is easily seen that the method described above yields, for all $\varepsilon > 0$, a positive measure μ of the form $\mu = \delta_0 + \nu$ such that $\| \nu \| < 1 + \varepsilon$ and $\{n \in \mathbf{N}; \hat{\mu}(n) = 0\}$ is a set of recurrence.

(2) The construction uses an argument to prove recurrence, different from the standard harmonic analysis argument. It is possible and perhaps of interest to give a more explicit description of the Poincaré sequence obtained.

(3) Following Ruzsa, call a family $\mathscr{C}$ of finite subset of $\mathbf{Z}$ *homogeneous* provided

(i) $A \in \mathscr{C}, B \subset A \Rightarrow B \in \mathscr{C}$,

(ii) $A \in \mathscr{C}, n \in \mathbf{Z} \Rightarrow A + n \in \mathscr{C}$.

Then the limit

$$d(\mathscr{C}) = \lim_{N \to \infty} \sup_{\substack{A \in \mathscr{C} \\ A \subset [0,N]}} \frac{|A|}{N}$$

exists and there is an infinite subset $\Lambda \subset \mathbf{Z}$ of density

$$d(\Lambda) = d(\mathscr{C})$$

which finite subsets belong to $\mathscr{C}$. This result is due to Ruzsa. An example of such a set is obtained by taking

$$\mathscr{C} = \{A \subset \mathbf{Z}, A \text{ finite} \mid A - A \subset \operatorname{supp} \hat{\mu}\}.$$

If $d(\mathscr{C}) > 0$, then $\hat{\mu}$ does not vanish on the difference set of a set of integers with positive denisty. Hence, in order that the complement of supp $\hat{\mu}$ should be (P), $d(\mathscr{C})$ has to be zero. Thus the problem considered in this paper is of a finite nature.

(4) A subset Λ of $\mathbf{Z}$ is called a *Sidon set* provided the $\mathscr{C}(G)$ and $A(\mathrm{T})$ norms are equivalent on trigonometric polynomials with Fourier transform supported by Λ. If Λ is a Sidon set, then Λ is not (v.d.C), since

$$\hat{\mu}(n) = -1, \quad n \in \Lambda \setminus \{0\}$$

for some positive measure μ (see [L–R]). The problem whether Λ may be approximative is open.

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